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On a stochastic delay difference equation with boundary conditions and its Markov property

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Abstract

In the present paper we consider the one-dimensional stochastic delay difference equation with boundary condition

$$\begin{cases} X_{n+1} = X_n + f(X_n) + g(X_{n-1}) + \xi_n, \\ X_0 = \psi(X_N), \end{cases}$$

$n \in \{0,\dots,N-1\}$, $N \geqslant 8$ (where $g(X_{-1}) \equiv 0$) . We prove that under monotonicity (or Lipschitz) conditions over the coefficients f, g and ψ, there exists a unique solution $\{Z_1,\dots,Z_N\}$ for this problem and we study its Markov property. The main result that we are able to prove is that the two-dimensional process $\{(Z_n, Z_{n+1}), 1 \leqslant n \leqslant N-1\}$ is a reciprocal Markov chain if and only if both the functions f and g are affine.

Keywords: Stochastic delay difference equation; Reciprocal Markov chain

1. Introduction

In the last five years several authors have studied, with different techniques, stochastic differential equations with boundary conditions of the following type:

$$\begin{cases} \mathrm{d}X_t = f(X_t)\,\mathrm{d}t + \sigma(X_t) \circ \mathrm{d}W_t, \quad t \in [0,1], \\ h(X_0, X_1) = 0 \end{cases} \tag{1.1}$$

(see Ocone and Pardoux, 1989; Nualart and Pardoux, 1991; Donati-Martin, 1991; Alabert et al., 1994). Due to the boundary condition, we cannot in general expect the solution to this type of equation to be adapted to the Wiener filtration. Therefore, in the study of Eq. (1.1) one makes use of the extended stochastic calculus for anticipating processes recently developed by several authors (see e.g. Nualart and Pardoux, 1988). A common result of these papers is that the solution is a Markov field (or a reciprocal process) if and only if the coefficients have some particular form. When

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$\sigma \equiv 1$, a nice dichotomy holds in the one-dimensional case (see Nualart and Pardoux, 1991): we have that the solution is a Markov field if and only if f is affine. This first result with constant diffusion in the scalar case has been generalized in the case where $\sigma(\cdot)$ is linear (Donati-Martin, 1991) or strictly positive (Alabert et al., 1994) and one proves that the Markov property of the unique solution to Eq. (1.1) is equivalent to the following condition over the coefficients:

$$f(x) = A\ \sigma(x) + B\ \sigma(x) \int_c^x \frac{1}{\sigma(t)}\,\mathrm{d}t$$

(where A, B and c are constants). In dimension higher than one, similar nice characterizations do not hold and one can prove (see Nualart and Pardoux, 1991; Ferrante, 1993; Ferrante and Nualart, 1995) that in some particular cases the Markov field property of the solution holds for coefficients that are partially free of any constraint.

At the same time several authors (see e.g. Donati-Martin, 1993; Alabert and Nualart, 1992; Ferrante and Nualart, 1995) have considered the discrete-time equivalent to the boundary value problem (1.1), that can be described by the following stochastic difference equation:

$$\begin{cases} X_{n+1} = X_n + f(X_n) + \sigma(X_n)\ \xi_n, & n \in \{0,\ldots,N-1\}, \\ X_0 = \psi(X_N). \end{cases} \tag{1.2}$$

Eq. (1.2) can be regarded as a discretization of Eq. (1.1) and in this sense the study of its Markov property helps to understand the continuous-time case. Eq. (1.2) has been studied in the one-dimensional case with $\sigma \equiv 1$ (see Donati-Martin, 1993): again one shows that the solution is a Markov field if and only if f is an affine mapping. This first result has been generalized in Ferrante and Nualart (1994), always in the scalar case, to the case where f and σ are increasing strictly positive mappings and the boundary condition is the linear equation $F_0X_0 + X_N = F$. One proves that the Markov property of the unique solution to Eq. (1.2) is equivalent to the following condition over the coefficients:

$$\begin{cases} x + f(x) = \beta\ x^{\gamma}, \text{ and} \\ \sigma(x) = \alpha\ x^{\gamma}, \qquad \text{for all } x \in \left[0, T^{-1}(T^{-1}(F))\right], \end{cases}$$

with $\alpha > 0$, $\beta > 0$, $0 < \gamma \leqslant 1$ and where $T(x) := x + f(x)$. As in the continuous-time case, the multidimensional problem is still not investigated, but one does not expect to obtain nice dichotomies as the previous ones.

A first step in the analysis of the multidimensional case could be the study of the following delay stochastic difference equation:

$$\begin{cases} X_{n+1} = X_n + f(X_n) + g(X_{n-1}) + \xi_n, \\ X_0 = \psi(X_N), \end{cases} \tag{1.3}$$

$n \in \{0,\ldots,N-1\}$, $N \geqslant 8$ (where $g(X_{-1}) \equiv 0$). This problem can be considered as a Êtrait-d'unionË between the one- and the two-dimensional cases. In fact, the technique that we use is the same as in the multidimensional case, but the result that we obtain is again a strong dichotomy as in the scalar case. Moreover, this problem could be thought of as the discretization of a similar continuous-time problem, not yet investigated.

In the second section we shall give two existence and uniqueness results for the problem (1.3). Moreover, we are able to prove that, under suitable regularity assumptions over the noise process $\{\xi_i,\ 0\leqslant i\leqslant N-1\}$, the solution of Eq. (1.3) $\{X_i, 1\leqslant i\leqslant N\}$ has an absolutely continuous law, that we shall compute explicitly.

In the third section we shall investigate the Markov property of the unique solution to Eq. (1.3), $\{X_i, 1\leqslant i\leqslant N\}$. A first difference, with respect to the classical one-dimensional problem, is that here it makes sense to require the Markov property just for the two-dimensional process $\{(X_i,X_{i+1}),\ 1\leqslant i\leqslant N-1\}$. The main result of this paper provides a complete characterization of the coefficients for which the Markov property holds. In fact we obtain that the Markov property holds if and only if both the coefficients f and g in (1.3) are affine maps.

To conclude this introduction, let us recall the definition of reciprocal Markov chain:

Definition 1.1. We shall say that a sequence of random variables $\{X_0,\dots,X_M\}$ is a *reciprocal Markov chain* if for every $0\leqslant m < n-1 < M-1$, the σ-fields $\sigma(X_m,\dots,X_n)$ and $\sigma(X_0,\dots,X_m,X_n,\dots,X_M)$ are conditionally independent given $\sigma(X_m,X_n)$.

2. Existence, uniqueness and absolute continuity

We shall consider in the present paper the following stochastic delay difference equation with nonlinear boundary condition

$$\begin{cases} X_{n+1} = X_n + f(X_n) + g(X_{n-1}) + \xi_n, & n\in\{0,\dots,N-1\}, \\ X_0 = \psi(X_N) \end{cases} \tag{2.1}$$

(with the convention that $g(X_{-1})\equiv 0$) where f, g and ψ are maps from $\mathbb{R}$ into itself and $\{\xi_i,\ 0\leqslant i\leqslant N-1\}$ is a sequence of independent random variables.

To deduce existence and uniqueness of solution for our equation, we shall follow two different approaches. The first one (in the spirit of Ferrante and Nualart, 1994) will require monotonicity conditions over the coefficients f, g and ψ, while the second one (that follows the ideas of Nualart and Pardoux (1988) and Donati-Martin (1993)) requires Lipschitz conditions.

Let us start by assuming the following set of conditions:

$$\begin{cases} \text{(i) } f \text{ is continuous and } x\longmapsto x+f(x) \text{ is increasing and onto } \mathbb{R}; \\ \text{(ii) } g \text{ is continuous and increasing;} \\ \text{(iii) } \psi \text{ is continuous and decreasing.} \end{cases} \tag{H.1}$$

Our first result is the following.

Proposition 2.1. *Under* (H.1), Eq. (2.1) *admits a unique solution.*

Proof. It is enough to prove that Eq. (2.1) admits a unique solution for each $\xi_0,\ldots,\xi_{N-1}$ fixed. Solving the first equation in (2.1) with initial data x_0 fixed, we have that, for all $n \in \{1,\ldots,N\}$, X_n is a function of x_0. Now, by (H.1(i)) we have that the map $x_0 \longmapsto X_1(x_0) = x_0 + f(x_0) + \xi_0$ is continuous, increasing and onto $\mathbb{R}$. If we consider now the map $x_0 \longmapsto X_2(x_0) = X_1(x_0) + g(x_0) + f(X_1(x_0)) + \xi_1$ and we take into account (H.1(ii)), we immediately obtain that it is itself a continuous, increasing map and that it is onto $\mathbb{R}$. Repeating the same computation for each n we obtain that the map $x_0 \longmapsto X_N(x_0)$ is itself continuous, increasing and onto $\mathbb{R}$. Since by (H.1(iii)) ψ is continuous and decreasing, the equation $x = \psi(X_N(x))$ admits a unique solution $\bar{x}_0$. Therefore we obtain that Eq. (2.1) admits a unique solution that can be recursively computed by solving the first equation in (2.1) with initial data $x_0 = \bar{x}_0$. □

An alternative result of existence and uniqueness of solution to Eq. (2.1) can be obtained under Lipschitz conditions over f, g and ψ. More precisely, we shall consider the following assumption:

$$\begin{cases} Id + f,\ g \text{ and } \psi \text{ are Lipschitz maps with constants } M,\ L \text{ and } K, \text{ respectively,} \\ \text{and we have that } K\, a_N < 1, \quad \text{where} \\ a_N = 2^{-N-1}\left(M^2 + 4L\right)^{-1/2}\left[\left(M + \sqrt{M^2+4L}\right)^{N+1} - \left(M - \sqrt{M^2+4L}\right)^{N+1}\right]. \end{cases} \tag{H.2}$$

In this case the following result holds.

Proposition 2.2. *Under* (H.2), Eq. (2.1) *admits a unique solution.*

Proof. We shall prove again that (2.1) admits a unique solution for each $\xi_0,\ldots,\xi_{N-1}$ fixed. As before, it will be sufficient to prove that the map $x \longmapsto \psi(X_N(x))$ admits a unique solution and to do it we shall prove that it is a strict contraction of $\mathbb{R}$ into itself.

We want to prove that for each x and y in $\mathbb{R}$

$$\left|\psi(X_N(x)) - \psi(X_N(y))\right| \leqslant \Lambda \left|x - y\right|, \tag{2.2}$$

with $0 < \Lambda < 1$. By (H.2), we have that

$$\left|\psi(X_N(x)) - \psi(X_N(y))\right| \leqslant K \left|X_N(x) - X_N(y)\right|$$

and

$$\begin{aligned} \left|X_N(x) - X_N(y)\right| &= \Big|X_{N-1}(x) + g(X_{N-2}(x)) + f(X_{N-1}(x)) \\ &\quad - X_{N-1}(y) - g(X_{N-2}(y)) - f(X_{N-1}(y))\Big| \\ &\leqslant M\left|X_{N-1}(x) - X_{N-1}(y)\right| + L\left|X_{N-2}(x) - X_{N-2}(y)\right|. \end{aligned}$$

For each $i = 1,\ldots,N-1$, we have

$$\Big|X_N(x) - X_N(y)\Big| \leqslant a_i \Big|X_{N-i}(x) - X_{N-i}(y)\Big| + L\, a_{i-1} \Big|X_{N-i-1}(x) - X_{N-i-1}(y)\Big|,$$

having defined recursively, for each $i = 1,\ldots,N-1$,

$$a_{i+1} = Ma_i + La_{i-1}, \quad a_0 = 1, \quad a_1 = M. \tag{2.3}$$

Now we have

$$\begin{aligned}\Big|X_N(x) - X_N(y)\Big| &\leqslant a_{N-1} \Big|X_1(x) - X_1(y)\Big| + L\, a_{N-2} \Big|x - y\Big| \\ &\leqslant \Big[a_{N-1}\, M + L\, a_{N-2}\Big] \Big|x - y\Big| = a_N \Big|x - y\Big|,\end{aligned}$$

and therefore that

$$\Big|\psi(X_N(x)) - \psi(X_N(y))\Big| \leqslant K\, a_N \Big|x - y\Big|.$$

It is easy to prove that

$$a_N = 2^{-N-1}\Big(M^2 + 4L\Big)^{-1/2}\left[\Big(M + \sqrt{M^2 + 4L}\,\Big)^{N+1} - \Big(M - \sqrt{M^2 + 4L}\Big)^{N+1}\right]$$

and therefore, by (H.2), we have that

$$x \longmapsto \psi(X_N(x))$$

is a strict contraction. □

In the sequel we shall always assume that (H.1) is satisfied and under stronger regularity conditions we shall be able to compute the probability law of the unique solution $(X_1,\ldots,X_N)$ to Eq. (2.1). From now on we shall assume the further hypothesis

$$\begin{cases} \{\xi_0,\ldots,\xi_{N-1}\} \text{ are independent absolutely continuous random variables} \\ \text{with a.e. strictly positive densities } \lambda_0(\cdot),\ldots,\lambda_{N-1}(\cdot), \text{ respectively.} \end{cases} \tag{H.3}$$

We can prove the following result.

Proposition 2.3. *Let f, g and ψ be of class C^1 and let* (H.3) *hold. If $f' > -1$, $g' \geqslant 0$ and $\psi' \leqslant 0$, then the random vector $(X_1,\ldots,X_N)$, unique solution to Eq.* (2.1), *has an absolutely continuous law with density*

$$f_X(x_1,\ldots,x_N) = \prod_{i=0}^{N-1}\Big[\lambda_i(x_{i+1} - x_i - g(x_{i-1}) - f(x_i))\Big]\,\Big|\mathscr{J}(x_1,\ldots,x_N)\Big| \tag{2.4}$$

(*with the convention* $x_0 = \psi(x_N)$ *and* $g(x_{-1}) \equiv 0$), *where*

$$\mathscr{J}(x_1,\dots,x_N) = 1 - \Big[1 + f'(\psi(x_N))\Big]\ \psi'(x_N)\ A_1(x_1,\dots,x_N), \tag{2.5}$$

and $A_1(x_1,\dots,x_N)$ *is recursively defined by*

$$\begin{cases} A_N = 1, \quad A_{N-1}(x_{N-1}) = 1 + f'(x_{N-1}), \\[2mm] A_n(x_n,\dots,x_{N-1}) = \big[1 + f'(x_n)\big]\, A_{n+1} + g'(x_n) A_{n+2} \quad \textit{for } 2 \leqslant n \leqslant N-2, \\[2mm] A_1(x_1,\dots,x_N) = \left[1 + f'(x_1) + \dfrac{g'(\psi(x_N))}{1 + f'(\psi(x_N))}\right] A_2 + g'(x_1) A_3. \end{cases}$$

Proof. Let us define the following map:

$$\Theta : \mathbb{R}^N \longrightarrow \mathbb{R}^N,$$

$$(\ \xi_0,\dots,\xi_{N-1}\ ) \longmapsto (\ X_1,\dots,X_N\ ).$$

Since, by the assumptions and (H.1), (2.1) admits a unique solution, the map Θ is well defined. Moreover, it is immediate to see that Θ is a bijection of $\mathbb{R}^N$ into itself and, by the smoothness of f, g and ψ, that it is a C^1-diffeomorphism. From (2.1) we have

$$\begin{cases} \xi_0 = \Theta_1^{-1}(x_1,\dots,x_N) = x_1 - \psi(x_N) - f(\psi(x_N)), \\[2mm] \xi_1 = \Theta_2^{-1}(x_1,\dots,x_N) = x_2 - x_1 - g(\psi(x_N)) - f(x_1), \\[2mm] \dots\ .\ \dots\dots\dots\dots\ .\ \dots\dots\dots\dots \\[2mm] \xi_{N-1} = \Theta_N^{-1}(x_1,\dots,x_N) = x_N - x_{N-1} - g(x_{N-2}) - f(x_{N-1}). \end{cases} \tag{2.6}$$

If we denote by $\mathscr{J}(x_1,\dots,x_N)$ the Jacobian of Θ^{-1}, it is easy to prove that the random variable $X = (X_1,\dots,X_N)$ has an absolutely continuous law with density

$$f_X(x_1,\dots,x_N) = f_\xi\left(\Theta^{-1}(x_1,\dots,x_N)\right)\ |\mathscr{J}(x_1,\dots,x_N)|,$$

where f_ξ denotes the density of the random vector $\xi = (\xi_0,\dots,\xi_{N-1})$. From (2.6) and (H.3), we have that

$$f_\xi\left(\Theta^{-1}(x_1,\dots,x_N)\right) = \prod_{i=1}^{N-1}\Big[\lambda_i(x_{i+1} - x_i - g(x_{i-1}) - f(x_i))\Big],$$

again with the convention $x_0 = \psi(x_N)$ and $g(x_{-1}) \equiv 0$ and to complete the proof it remains to compute the Jacobian of Θ^{-1}. From (2.6) we have that $\mathscr{J}(x_1,\dots,x_N)$ is equal to the determinant of the following matrix:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -[1+f'(\psi(x_N))]\psi'(x_N) \\ -1-f'(x_1) & 1 & \cdots & 0 & -g'(\psi(x_N))\psi'(x_N) \\ -g'(x_1) & -1-f'(x_2) & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots -1-f'(x_{N-1}) & & 1 \end{bmatrix} . \quad (2.7)$$

Recalling that $1+f'(x)>0,\ \forall x \in \mathbb{R}$, it holds

$$\mathscr{J}(x_1,\ldots,x_N) = \det B,$$

where B is the following matrix:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -[1+f'(\psi(x_N))]\psi'(x_N) \\ -\dfrac{g'(\psi(x_N))}{1+f'(\psi(x_N))} - 1 - f'(x_1) & 1 & \cdots & 0 & 0 \\ -g'(x_1) & -1-f'(x_2) & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots -1-f'(x_{N-1}) & & 1 \end{bmatrix} .$$

Expanding now the determinant of B by means of minors of the first row, we obtain

$$\mathscr{J}(x_1,\ldots,x_N) = 1 - (-1)^{N+1}\ \psi'(x_N)\ \left[1+f'(\psi(x_N))\right]\ \det C,$$

where C is the following $(N-1)\times(N-1)$ matrix:

$$\begin{bmatrix} -\dfrac{g'(\psi(x_N))}{1+f'(\psi(x_N))} - 1 - f'(x_1) 1 \cdots & 0 & 0 & 0 \\ . & \cdots . & . & . \\ 0 & 0 \cdots -1-f'(x_{N-3}) & 1 & 0 \\ 0 & 0 \cdots \quad -g'(x_{N-3}) & -1-f'(x_{N-2}) & 1 \\ 0 & 0 \cdots \quad 0 & -g'(x_{N-2}) & -1-f'(x_{N-1}) \end{bmatrix} .$$

Defining recursively

$$\begin{cases} A_N = 1, \quad A_{N-1}(x_{N-1}) = 1 + f'(x_{N-1}), \\ A_n(x_n,\dots,x_{N-1}) = \left[1 + f'(x_n)\right] A_{n+1} + g'(x_n) A_{n+2} \quad \text{for} \quad 2 \leqslant n \leqslant N-2, \\ A_1(x_1,\dots,x_N) = \left[1 + f'(x_1) + \dfrac{g'(\psi(x_N))}{1 + f'(\psi(x_N))}\right] A_2 + g'(x_1) A_3 \end{cases}$$

(notice that the assumptions over f, g and ψ imply that

$$A_n > 0 \tag{2.8}$$

for every $n = 1,\dots,N$), a simple computation shows that

$$\det C = (-1)^{N-1} A_1.$$

At the end we obtain that

$$\mathscr{J}(x_1,\dots,x_N) = 1 - \left[1 + f'(\psi(x_N))\right] \psi'(x_N)\, A_1(x_1,\dots,x_N),$$

and the proof is complete. □

3. Markov property

We now want to study the Markov property of the unique solution to Eq. (2.1). First of all we shall recall a simple result (see Ferrante and Nualart, 1994) that allows us to give a characterization of the reciprocal Markov chain property of a random vector which has an absolutely continuous law.

Lemma 3.1. *Let us assume that the vector* $X = (X_0,\dots,X_M)$ *has an absolutely continuous law with density* $f_0(x_0,\dots,x_M)$. *Then* X *is a reciprocal Markov chain if and only if, for every* $0 \leqslant m < n-1 < M-1$, *there exist two measurable functions* $f_1(x_m,\dots,x_n)$ *and* $f_2(x_0,\dots,x_m,x_n,\dots,x_M)$ *such that*

$$f_0(x_0,\dots,x_M) = f_1(x_m,\dots,x_n)\; f_2(x_0,\dots,x_m,x_n,\dots,x_M) \quad a.e.$$

An easy application of the previous lemma gives the following result:

Proposition 3.1. *Under* (H.3) *and assuming that* f, g *and* ψ *are maps of class* C^1 *such that* $f' > -1$, $g' \geqslant 0$ *and* $\psi' \leqslant 0$, *the two-dimensional process* $\{(X_n,X_{n+1}),\ 1 \leqslant n \leqslant N-1\}$, *where* $\{X_n,\ 1 \leqslant n \leqslant N\}$ *denotes the unique solution of Eq.* (2.1), *is a reciprocal Markov chain if and only if for each* $1 \leqslant m < n-2 < N-3$ *there exist two measurable functions*

$$\Phi_1 : (0,+\infty)^{n-m+2} \longrightarrow \mathbb{R},$$

$$\Phi_2 : (0,+\infty)^{N-n+m+2} \longrightarrow \mathbb{R}$$

such that

$$\begin{aligned} 1 - \left[1 + f'(\psi(x_N))\right] \ \psi'(x_N)\, A_1(x_1,\ldots,x_N) \\ = \ \Phi_1\,(x_m,\ldots,x_{n+1})\ \ \Phi_2\,(x_1,\ldots,x_{m+1},x_n,\ldots,x_N) \quad a.e. \end{aligned} \tag{3.1}$$

Proof. It follows immediately from Lemma 3.1 and (2.4)–(2.5). □

In the sequel we shall need this simple technical lemma.

Lemma 3.2. *Let F be a twice continuously differentiable and positive real function defined on $\mathbb{R}^{\alpha+\beta}$, where α and β are positive integers. The following two statements are equivalent*:

(1) *There exist two measurable functions ϕ_1 and ϕ_2 such that*

$$1 + F(x_1,x_2) \ = \ \phi_1\,(x_1)\ \phi_2\,(x_2) \qquad \textit{for all } x_1 \in \mathbb{R}^{\alpha},\ x_2 \in \mathbb{R}^{\beta}.$$

(2) *We have*

$$\left[1 + F(x_1,x_2)\right]\ \frac{\partial^2}{\partial x_1^i\,\partial x_2^j}F(x_1,x_2) - \frac{\partial}{\partial x_1^i}F(x_1,x_2)\ \frac{\partial}{\partial x_2^j}F(x_1,x_2) = 0$$

for all $i \in \{1,\ldots,\alpha\}$, $j \in \{1,\ldots,\beta\}$ and for every $(x_1,x_2) \in \mathbb{R}^{\alpha+\beta}$.

Proof. From the regularity of the function $F(x_1,x_2)$, we obtain that $\phi_1\,(x_1)$ and $\phi_2\,(x_2)$ have to be themselves regular. Now, taking the logarithm in (1) (we have that $1 + F(x_1,x_2)$ is strictly positive) and differentiating with respect to x_1^i and x_2^j, we immediately obtain (2). Integrating (2) with respect to x_1^i and x_2^j one obtains easily the converse result. □

Remark 3.1. In Alabert and Nualart (1992) and Ferrante and Nualart (1994) one makes use of a stronger technical lemma (see Alabert and Nualart, 1992, Lemma 2.3), since in those papers one can assume that the function $F(x_1,x_2)$ factorizes as a product of two functions $G_1(x_1)$ and $G_2(x_2)$. Here, due to the factor $A_1(x_1,\ldots,x_N)$, we have to use Lemma 3.2. This lack of factorization is a characteristic of the multi-dimensional case and makes the analysis in this paper more complicated than in the case of Ferrante and Nualart (1994).

Making use of the factorization property of Proposition 3.1, the technical Lemma 3.2 and requiring the strict monotonicity of the map g, we are now able to prove the main result of the present paper.

Theorem 3.1. *Let $N \geqslant 8$ and let us assume that* (H.3) *holds, f, g and ψ are of class C^2, with $f' > -1$, $g' > 0$, $\psi' \leqslant 0$ and $\psi' \not\equiv 0$. The two-dimensional process $\{(X_n,X_{n+1}), 1 \leqslant n \leqslant N-1\}$, associated to the unique solution of Eq.* (2.1) *$\{X_n, 1 \leqslant n \leqslant N\}$, is a reciprocal Markov chain if and only if both the functions f and g are affine.*

Remark 3.2. Note that if $\psi' \equiv 0$, then X_0 is deterministic and the two-dimensional process $\{(X_n, X_{n+1}), 1 \leqslant n \leqslant N-1\}$ is a Markov chain for each pair of coefficients f and g.

Proof of Theorem 3.1. By Proposition 3.1, it is sufficient to prove that condition (3.1) holds for each $1 \leqslant m < n-2 < N-3$ and for suitable measurable functions

$$\Phi_1 : (0,+\infty)^{n-m+2} \longrightarrow \mathbb{R} \quad \text{and} \quad \Phi_2 : (0,+\infty)^{N-n+m+2} \longrightarrow \mathbb{R}$$

if and only if both the functions f and g are affine.

Sufficiency: Let us assume that f and g are affine maps. In this case we have

$$f'(\cdot) = k_1 \quad \text{and} \quad g'(\cdot) = k_2,$$

where k_1 and k_2 are suitable constants, and therefore that $A_1(x_1,\ldots,x_N)$ is itself a constant. We have

$$1 - \Big[1 + f'(\psi(x_N))\Big]\ \psi'(x_N)\ A_1(x_1,\ldots,x_N) = 1 - \Big[1 + k_1\Big]\ \psi'(x_N)\ A_1$$

and therefore (3.1) trivially holds by taking

$$\Phi_1 \equiv 1 \quad \text{and} \quad \Phi_2(x_1,\ldots,x_{m+1},x_n,\ldots,x_N) = 1 - \Big[1 + k_1\Big]\ \psi'(x_N)\ A_1.$$

Necessity: Let us now assume that for each $1 \leqslant m < n-2 < N-3$ there exist two measurable functions

$$\Phi_1 : (0,+\infty)^{n-m+2} \longrightarrow \mathbb{R} \quad \text{and} \quad \Phi_2 : (0,+\infty)^{N-n+m+2} \longrightarrow \mathbb{R}$$

such that

$$\begin{aligned} &1 - \big[1 + f'(\psi(x_N))\big]\ \psi'(x_N)\ A_1(x_1,\ldots,x_N) \\ &\quad = \Phi_1(x_m,\ldots,x_{n+1})\ \Phi_2(x_1,\ldots,x_{m+1},x_n,\ldots,x_N) \quad \text{a.e.} \end{aligned}$$

To avoid the trivial cases, let us choose m and n such that

$$3 \leqslant m < n-2 < N-3$$

(in this way the interior and exterior σ-fields are not degenerate) and fix $i \in \{2,\ldots, m-1\}$ and $j \in \{m+2,\ldots,n-1\}$. We can apply Lemma 3.2 to the function

$$\Big(x_1,\ldots,x_{m-1},x_{m+2},\ldots,x_{n-1},x_{n+2},\ldots,x_N\Big) \longmapsto -\Big[1 + f'(\psi(x_N))\Big]\ \psi'(x_N)\ A_1(x_1,\ldots,x_N).$$

We obtain therefore that

$$\begin{aligned} &\Big[1 + f'(\psi(x_N))\Big]\ \psi'(x_N)\ \Big\{1 - \Big[1 + f'(\psi(x_N))\Big]\ \psi'(x_N)\ A_1\Big\}\ \frac{\partial^2}{\partial x_j \partial x_i} A_1 \\ &\quad + \Big[1 + f'(\psi(x_N))\Big]^2\ \Big(\psi'(x_N)\Big)^2\ \frac{\partial A_1}{\partial x_i}\ \frac{\partial A_1}{\partial x_j} = 0. \end{aligned} \tag{3.2}$$

Now, since $1 + f'(\psi(x_N)) > 0$ and $\psi' \not\equiv 0$, there exists $x_N \in \mathbb{R}$ such that from (3.2) we have

$$\left\{1 - \left[1 + f'(\psi(x_N))\right] \psi'(x_N)\, A_1\right\} \frac{\partial^2}{\partial x_j \partial x_i} A_1$$

$$+\left[1 + f'(\psi(x_N))\right] \psi'(x_N)\, \frac{\partial A_1}{\partial x_i}\, \frac{\partial A_1}{\partial x_j} = 0. \tag{3.3}$$

We now have to compute $\partial A_1/\partial x_i$ and $(\partial^2/\partial x_j\, \partial x_i)A_1$. It is not difficult to prove that for each $1 < j < N-1$

$$\frac{\partial A_1}{\partial x_j} = B_{j-1}(x_1,\dots,x_{j-1},x_N) \left[f''(x_j)A_{j+1}(x_{j+1},\dots,x_{N-1})\right.$$

$$\left.+g''(x_j)A_{j+2}(x_{j+2},\dots,x_{N-1})\right], \tag{3.4}$$

where the B_j's are recursively defined by

$$\begin{cases} B_0 = 1 \\ B_1 = B_1(x_1,x_N) = \dfrac{g'(\psi(x_N))}{1 + f'(\psi(x_N))} + 1 + f'(x_1), \\ B_j = B_j(x_1,\dots,x_j,x_N) = [1 + f'(x_j)]B_{j-1} + g'(x_{j-1})B_{j-2}, \quad \text{for } j \geqslant 2. \end{cases} \tag{3.5}$$

Moreover, we have

$$\frac{\partial A_{i+1}}{\partial x_j} = \tilde{B}^{i+1}_{j-1}(x_{i+1},\dots,x_{j-1}) \left[f''(x_j)A_{j+1}(x_{j+1},\dots,x_{N-1})\right.$$

$$\left.+g''(x_j)A_{j+2}(x_{j+2},\dots,x_{N-1})\right], \tag{3.6}$$

for each $j \geqslant i+2$, where

$$\begin{cases} \tilde{B}^{i+1}_{i} = 1, \\ \tilde{B}^{i+1}_{i+1}(x_{i+1}) = 1 + f'(x_{i+1}), \\ \tilde{B}^{i+1}_{j}(x_{i+1},\dots,x_j) = [1 + f'(x_j)]\tilde{B}^{i+1}_{j-1} + g'(x_{j-1})\tilde{B}^{i+1}_{j-2}, \quad \text{for } j \geqslant i+2. \end{cases} \tag{3.7}$$

Remark 3.3. Notice that, under the present assumptions, we have that B_j and $\tilde{B}^{i+1}_j$ are strictly positive for every $i + 2 \leqslant j$.

From (3.4) we obtain that (3.3) is equal to the following equation,

$$\Big\{1-\Big[1+f'(\psi(x_N))\Big]\ \psi'(x_N)\ A_1\Big\}\ \frac{\partial}{\partial x_j}\ \Big[B_{i-1}(x_1,\dots,x_{i-1},x_N)\ \Big(f''(x_i)A_{i+1}$$

$$+g''(x_i)A_{i+2}\Big)\Big]+\Big[1+f'(\psi(x_N))\Big]\ \psi'(x_N)\ B_{i-1}(x_1,\dots,x_{i-1},x_N)\ \Big[f''(x_i)A_{i+1}$$

$$+g''(x_i)A_{i+2}\Big]\ B_{j-1}(x_1,\dots,x_{j-1},x_N)\ \Big[f''(x_j)A_{j+1}+g''(x_j)A_{j+2}\Big]=0,$$

and by (3.6) that

$$\Big\{1-\Big[1+f'(\psi(x_N))\Big]\psi'(x_N)\ A_1\Big\}\ B_{i-1}\ \Big[f''(x_i)\ \tilde{B}^{i+1}_{j-1}\Big(f''(x_j)A_{j+1}+g''(x_j)A_{j+2}\Big)$$

$$+\ g''(x_i)\ \tilde{B}^{i+2}_{j-1}\ \Big(f''(x_j)A_{j+1}+g''(x_j)A_{j+2}\Big)\Big]\ +\ \Big[1+f'(\psi(x_N))\Big]$$

$$\times\ \psi'(x_N)\ B_{i-1}\ \Big[f''(x_i)A_{i+1}+g''(x_i)A_{i+2}\Big]\ B_{j-1}\Big[f''(x_j)A_{j+1}+g''(x_j)A_{j+2}\Big]=0. \tag{3.8}$$

Let us assume that f or g is not an affine map and let us prove that this assumption leads to a contradiction. We shall need the following technical lemma:

Lemma 3.3. *Under the assumptions of Theorem* 3.1, *if f or g is nonlinear, then there exists U, an open and nonempty subset of $\mathbb{R}$, such that*

(a) $(\partial A_i/\partial x_i)(x_i,\dots,x_{N-1})\neq 0$ *for a.e.* $(x_i,\dots,x_{N-1})\in U^{N-i}$ *and* $i\in\{2,\dots,N-3\}$,

(b) $\Big(f''(x)\Big)^2+\Big(g''(x)\Big)^2\ \neq\ 0$ *on* U.

Proof.

Step 1: Let f be affine and g not (and the same holds when g is affine and f not); by the regularity conditions over g there exists an open, nonempty subset U of $\mathbb{R}$ where $g''\neq 0$. For $i\in\{2,\dots,N-3\}$, we shall have that

$$\frac{\partial A_i}{\partial x_i}=g''(x_i)A_{i+2}.$$

Now, since $A_{i+2}>0$, the result is proved, as $\partial A_i/\partial x_i\neq 0$ on U^{N-i}, for $i\in\{2,\dots,N-3\}$.

Step 2: Let f and g be both nonlinear; by the regularity of f and g there will exist two open subsets of $\mathbb{R}$, U and V such that $f''\neq 0$ on U and $g''\neq 0$ on V, respectively. We shall proceed by induction, proving the property (a) also for $i=N-2,\ N-1$.

Let us start by A_{N-1}; since

$$\frac{\partial A_{N-1}}{\partial x_{N-1}}(x_{N-1})=f''(x_{N-1}),$$

we have that condition (a) holds on U. Let us now consider A_{N-2}; differentiating with respect to x_{N-2} and x_{N-1}, we obtain that

$$\frac{\partial}{\partial x_{N-1}}\Big(\frac{\partial A_{N-2}}{\partial x_{N-2}}(x_{N-2},x_{N-1})\Big) = f''(x_{N-2})\ \frac{\partial A_{N-1}}{\partial x_{N-1}}(x_{N-1})\ \neq\ 0$$

for every $(x_{N-2},x_{N-1}) \in U^2$. It clearly implies that

$$\frac{\partial A_{N-2}}{\partial x_{N-2}}(x_{N-2},x_{N-1}) \neq 0 \quad \text{for a.e. } (x_{N-2},x_{N-1}) \in U^2$$

and condition (a) is proved to be true.

Let us now assume that (a) holds for every $j \in \{i+1,\dots,N-1\}$ and prove that it holds for $j = i$. Differentiating now A_i with respect to x_i and to x_{i+1}, we obtain that

$$\frac{\partial}{\partial x_{i+1}}\Big(\frac{\partial A_i}{\partial x_i}(x_i,\dots,x_{N-1})\Big) = f''(x_i)\ \frac{\partial A_{i+1}}{\partial x_{i+1}}(x_{i+1},\dots,x_{N-1})\ \neq\ 0$$

for a.e. $(x_i,\dots,x_{N-1}) \in U^{N-i}$, by the induction assumption. Therefore,

$$\frac{\partial A_i}{\partial x_i}(x_i,\dots,x_{N-1}) \neq 0 \quad \text{for a.e. } (x_i,\dots,x_{N-1}) \in U^{N-i},$$

and condition (a) holds. To complete the proof it will be sufficient to recall that (b) is satisfied on U. $\square$

From now on we shall assume that $x_2,\dots,\ x_{N-1}$ belong to U, the open set defined in Lemma 3.3. Since

$$f''(x_j)A_{j+1} + g''(x_j)A_{j+2} \ =\ \frac{\partial A_j}{\partial x_j} \neq 0 \quad \text{a.e. on} \quad U^{N-j},$$

Eq. (3.8) is equivalent to the following one:

$$\Big\{1 - [1 + f'(\psi(x_N))]\ \psi'(x_N)\ A_1\Big\}\ \Big[f''(x_i)\tilde{B}^{i+1}_{j-1} + g''(x_i)\tilde{B}^{i+2}_{j-1}\Big]$$

$$+\ \Big[1 + f'(\psi(x_N))\Big]\ \psi'(x_N)\ B_{j-1}\ \Big[f''(x_i)A_{i+1} + g''(x_i)A_{i+2}\Big] = 0. \tag{3.9}$$

Differentiating now with respect to x_{j+1} we have

$$\begin{aligned}
&-\Big[1 + f'(\psi(x_N))\Big]\ \psi'(x_N)\ \Big[f''(x_i)\tilde{B}^{i+1}_{j-1} + g''(x_i)\tilde{B}^{i+2}_{j-1}\Big] \\
&B_j\ \Big[f''(x_{j+1})A_{j+2} + g''(x_{j+1})A_{j+3}\Big] \\
&+\Big[1 + f'(\psi(x_N))\Big]\ \psi'(x_N)\ B_{j-1}\ \Big[f''(x_i)\tilde{B}^{i+1}_{j} + g''(x_i)\tilde{B}^{i+2}_{j}\Big] \\
&\times\ \Big[f''(x_{j+1})A_{j+2} + g''(x_{j+1})A_{j+3}\Big] = 0.
\end{aligned}$$

Since

$$[1+f'(\psi(x_N))]\ \psi'(x_N)\ \neq\ 0$$

and, by Lemma 3.3,

$$f''(x_{j+1})\ A_{j+2}+g''(x_{j+1})\ A_{j+3}=\frac{\partial A_{j+1}}{\partial x_{j+1}}\neq 0 \quad \text{a.e.},$$

we obtain

$$B_j\ \left[f''(x_i)\tilde{B}_{j-1}^{i+1}\ +\ g''(x_i)\tilde{B}_{j-1}^{i+2}\right]=B_{j-1}\ \left[f''(x_i)\tilde{B}_{j}^{i+1}\ +\ g''(x_i)\tilde{B}_{j}^{i+2}\right]. \tag{3.10}$$

Now, recalling that

$$\tilde{B}_j^{i+1}=[1+f'(x_j)]\tilde{B}_{j-1}^{i+1}+g'(x_{j-1})\tilde{B}_{j-2}^{i+1},$$

$$\tilde{B}_j^{i+2}=[1+f'(x_j)]\tilde{B}_{j-1}^{i+2}+g'(x_{j-1})\tilde{B}_{j-2}^{i+2},$$

$$B_j=[1+f'(x_j)]B_{j-1}\ +\ g'(x_{j-1})B_{j-2},$$

from (3.10) we have

$$B_{j-1}\ \left[f''(x_i)\tilde{B}_{j-2}^{i+1}\ +\ g''(x_i)\tilde{B}_{j-2}^{i+2}\right]=B_{j-2}\ \left[f''(x_i)\tilde{B}_{j-1}^{i+1}\ +\ g''(x_i)\tilde{B}_{j-1}^{i+2}\right].$$

Proceeding in the same way, we obtain at the end that

$$B_{i+2}\ \left[f''(x_i)\tilde{B}_{i+1}^{i+1}\ +\ g''(x_i)\tilde{B}_{i+1}^{i+2}\right]\ =\ B_{i+1}\ \left[f''(x_i)\tilde{B}_{i+2}^{i+1}\ +\ g''(x_i)\tilde{B}_{i+2}^{i+2}\right]. \tag{3.11}$$

A simple computation gives that (3.11) is equivalent to

$$B_{i+1}\ f''(x_i)=B_i\ \left[f''(x_i)\ \left(1+f'(x_{i+1})\right)+g''(x_i)\right]$$

and therefore

$$f''(x_i)\ \{B_{i+1}-B_i\ [1+f'(x_{i+1})]\}=g''(x_i)B_i$$

which implies

$$B_{i-1}\ g'(x_i)\ f''(x_i)=B_i\ g''(x_i). \tag{3.12}$$

From (3.12) and the positivity of g' and $1+f'$, we deduce

$$\frac{\partial}{\partial x_i}\ \left(\log\ g'(x_i)\right)=\frac{\partial}{\partial x_i}\ (\log\ B_i)$$

which gives that

$$1+f'(x_i)=\frac{K_1}{B_{i-1}}\ g'(x_i)-\frac{B_{i-2}}{B_{i-1}}\ g'(x_{i-1}), \tag{3.13}$$

with K_1 a strictly positive function.

It is easy to see that if the function $K_1(x_1,\dots,x_{i-1},x_N)/B_{i-1}(x_1,\dots,x_{i-1},x_N)$ is not constant, then both f and g have to be affine functions on U, which leads to a contradiction with our hypothesis. Assuming therefore that $K_1/B_{i-1} = a > 0$ on U^i, from (3.13) we have that there exists a positive constant b such that

$$\frac{B_{i-2}(x_1,\ldots,x_{i-2},x_N)}{B_{i-1}(x_1,\ldots,x_{i-1},x_N)}\, g'(x_{i-1}) = b \tag{3.14}$$

and therefore we have that

$$1 + f'(x) = a\, g'(x) - b \quad \text{for every } x \in U. \tag{3.15}$$

From the definition of B_{i-1} and (3.14), we obtain

$$B_{i-2}\, g'(x_{i-1}) = b\left[B_{i-2}\Big(1 + f'(x_{i-1})\Big) + B_{i-3}\, g'(x_{i-2})\right],$$

and, since $x_{i-1} \in U$, by (3.15) we have

$$B_{i-2}\, g'(x_{i-1})\Big(1 - ab\Big) = b\left[-\; B_{i-2}\, b + B_{i-3}\, g'(x_{i-2})\right]. \tag{3.16}$$

If $1 - ab \neq 0$, we obtain that $g'(x_{i-1})$ is constant on U and again we obtain a contradiction with our assumption. If $1 - ab = 0$ we shall arrive at a contradiction. In fact from (3.16)

$$\frac{B_{i-3}(x_1,\ldots,x_{i-3},x_N)}{B_{i-2}(x_1,\ldots,x_{i-2},x_N)}\, g'(x_{i-2}) = b,$$

and proceeding in the same way for every i at the end we obtain that

$$\frac{B_0}{B_1(x_1,x_N)}\, g'(x_1) = b\ . \tag{3.17}$$

Recalling that $B_0 = 1$,

$$B_1(x_1,x_N) = \frac{g'(\psi(x_N))}{1 + f'(\psi(x_N))} + 1 + f'(x_1)$$

and choosing $x_1 \in U$, from (3.15) we have

$$g'(\psi(x_N)) = b\,\Big(1 + f'(\psi(x_N))\Big) \quad \text{for every } x_N \in \mathbb{R}, \text{ with } \psi(x_N) \neq 0. \tag{3.18}$$

Choosing now $x_1 \in \mathrm{Im}(\psi)\backslash\{0\}$, from (3.18) it follows

$$g'(x_1) = b\,\Big(1 + f'(x_1)\Big). \tag{3.19}$$

From (3.17)–(3.19) we deduce

$$b^2 = 0,$$

which clearly leads to a contradiction.

Therefore, if the factorization property (3.1) holds, then f and g have to be affine maps. $\square$

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